C.B.S.E. CLASS – XII BOARD - 2014

MATHEMATICS

Series: OSR/1 **Code No. 65/1/3**

Time allowed 3 hours Maximum Marks: 100

General Instructions:

(i) All questions are compulsory.

(ii) Marks for each question are indicated against it.

(iii) Question numbers 1 to 10 are very short answer type questions of 1 mark each

(iv) Question numbers 11 to 22 are short answer type questions of 4 marks each.

(v) Question numbers 23 to 29 are short answer type question of 6 marks each

Section A

1. If $2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$, find (x – y) **[1 Marks]**

Sol. $\begin{bmatrix}6 & 8\\10 & 2x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}\Rightarrow\begin{bmatrix}7 & 8+y\\10 & 2x+1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$

Thus $x=2,\ y=-8$

2. Solve the following matrix equation for $x:,\begin{bmatrix}x & 1\end{bmatrix}\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix}=O$ **[1 Marks]**

Sol. $\begin{bmatrix}x & 1\end{bmatrix}\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix}=\begin{bmatrix}x\times1+1\times(-2) & x\times0+1\times0\end{bmatrix}$

$=\begin{bmatrix}x-2 & 0\end{bmatrix}$

Therefore x = 2

3. If $\begin{vmatrix}2x & 5\\8 & x\end{vmatrix}=\begin{vmatrix}6 & -2\\7 & 3\end{vmatrix}$, write the value of x. **[1 Marks]**

Sol. $2x^2-40=18+14$

$2x^2=72\Rightarrow x^2=36\Rightarrow x=\pm6$

4. Write the anti derivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ **[1 Marks]**

Sol. Anti derivative of $3\sqrt{x}+\frac{1}{\sqrt{x}}$

$=\frac{3x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}$

$=2\sqrt{x}(x+1)$

5. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x. **[1 Marks]**

Sol. $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

$\sin^{-1}\left(\frac{1}{5}\right)+\cos^{-1}(x)=\frac{\pi}{2}\Rightarrow\sin^{-1}\left(\frac{1}{5}\right)=\frac{\pi}{2}-\cos^{-1}x\Rightarrow\sin^{-1}\left(\frac{1}{5}\right)=\sin^{-1}(x)$

$x=\frac{1}{5}$

6. Let * be a binary operation, on the set of all non – zero real numbers, given by $a * b=\frac{ab}{5}$ for all $a,b\in R-\{0\}$. Find the value of x, given that $2*(x*5)=10$. **[1 Marks]**

Sol. $a * b = \frac{ab}{5}$

$2 * (x * 5) = 10 \Rightarrow 2 * \left(\frac{x}{5}\right) = 10$

$\frac{2x}{5} = 10 \Rightarrow x = 25$

7. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$. **[1 Marks]**

Sol. Let $a = \hat{i} + 3\hat{j} + 7\hat{k}, \vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a}.\vec{b}}{|b|} = \frac{2 - 9 + 42}{\sqrt{4 + 9 + 36}} = 5$

8. Write the vector equation of the plane, passing through the point (a, b, c) and parallel to the plane $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$. **[1 Marks]**

Sol. Equation of plane parallel to $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$ is $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = \lambda$

$\Rightarrow \left(a\hat{i} + b\hat{j} + c\hat{k}\right).\left(\hat{i} + \hat{j} + \hat{k}\right) = \lambda \Rightarrow \lambda = a + b + c$

$\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = a + b + c$

9. Evaluate : $\int_0^{\pi/2} e^x (\sin x - \cos x) dx$. **[1 Marks]**

Sol. $I = \int_0^{\pi/2} e^x (\sin x - \cos x) dx = -\int_0^{\pi/2} e^x (\cos x - \sin x) dx$

Using property

$\int e^x (f(x) + f'(x)) dx = e^x f(x) + c,$ we get

$I = -\left|e^x \cos x\right|_0^{\pi/2} = 1$

10. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$. **[1 Marks]**

Sol. $\vec{a} + \vec{b} = 4\hat{i} + 3\hat{j} + 12\hat{k}$

$\left(\vec{a} + \vec{b}\right) = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{\sqrt{16 + 9 + 144}} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{13}$

Section B

11. Prove that for any three vectors $\vec{a},\vec{b},\vec{c}$ **[4 Marks]**

$\left[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\vec{c}+\vec{a}\right]=2\left[\vec{a},\vec{b},\vec{c}\right]$

OR

Vectors $\vec{a},\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $\left|\vec{a}\right|=3,\ \left|\vec{b}\right|=5$ and $\left|\vec{c}\right|=7$. Find the angle between $\vec{a}$ and $\vec{b}$

Sol. $\left(\vec{a}+\vec{b}\right).\left[\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right]=\left(\vec{a}+\vec{b}\right).\left[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}\right]$

= 2 [a b c]

OR

$\vec{c}=\left(\vec{a}+\vec{b}\right)$

$\left|\vec{c}\right|=\left|\vec{a}+\vec{b}\right|$

$\left|\vec{c}\right|^2=\left|\vec{a}+\vec{b}\right|^2=\left(\vec{a}+\vec{b}\right).\left(\vec{a}+\vec{b}\right)$

$c^2=a^2+b^2+2\vec{a}.\vec{b}$

$49=9+25+30\cos\theta\Rightarrow\cos\theta=\dfrac{1}{2}\Rightarrow\theta=\dfrac{\pi}{3}$

12. Solve the following differential equation: **[4 Marks]**

$\left(x^2-1\right)\dfrac{dy}{dx}+2xy=\dfrac{2}{x^2-1}$

Sol. $\dfrac{dy}{dx}+\dfrac{2x}{x^2-1}y=\dfrac{2}{\left(x^2-1\right)^2}$

Integrating factor $=e^{\int\frac{2x}{x^2-1}dx}=e^{\ln\left(x^2-1\right)}=x^2-1$

$y\left(I.f\right)=\int Q\left(I.F\right)dx+C$

$y\left(x^2-1\right)=\int\dfrac{2}{\left(x^2-1\right)^2}\left(x^2-1\right)dx+c$

$y\left(x^2-1\right)=2\int\dfrac{1}{x^2-1}dx+c=\ln\left|\dfrac{x-1}{x+1}\right|+C$

13. Evaluate : $\displaystyle\int\frac{\sin^6x+\cos^6x}{\sin^2x.\cos^2x}dx$ **[4 Marks]**

Or

Evaluate : $\displaystyle\int\left(x-3\right)\sqrt{x^2+3x-18}\,dx$

Sol. $\displaystyle\int\frac{\sin^6x+\cos^6x}{\sin^2x\cos^2x}dx$

$\displaystyle=4\int\frac{\left(\sin^2x+\cos^2x\right)^3-3\sin^2x\cos^2x\left(\sin^2x+\cos^2x\right)}{4\sin^2x\cos^2x}dx$

$$= \int \frac{4-3\sin^2 2x}{\sin^2 2x} dx = 4\int \text{cosec}^2 2x\, dx - \int 3dx$$
$$= -2\cot 2x - 3x + c$$

OR

$$\int (x-3)\sqrt{x^2+3x-18}\, dx$$
Let $x-3 = A(2x+3)+B$

$2A = 1;\ 3A+B = -3$

$A = \frac{1}{2}$

$\Rightarrow \frac{3}{2} + B = -3$

$B = -3 - \frac{3}{2} = \frac{-9}{2}$

$$\int (x-3)\sqrt{x^2+3x-18}\, dx = \int \left\{\frac{1}{2}(2x+3) - \frac{9}{2}\right\}\sqrt{x^2+3x-18}\, dx$$
$$= \frac{1}{2}\int \sqrt{x^2+3x-18}\,(2x+3)\, dx - \frac{9}{2}\int \sqrt{x^2+3x-18}\, dx$$
$$= \frac{1}{2}.\frac{\left(x^2+3x-18\right)^{3/2}}{\frac{3}{2}} - \frac{9}{2}\sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}}\, dx$$
$$= \frac{1}{3}\left(x^2+3x-18\right)^{3/2} - \frac{9}{2}\left[\left(\frac{x+\frac{3}{2}}{2}\right)\sqrt{x^2+3x-18} + \frac{81}{8}\sin^{-1}\frac{x+3/2}{9/2}\right] + c$$

14. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is **[4 Marks]**

(a) strictly increasing

(b) strictly decreasing

OR

Find the equation of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

Sol. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f(x) = 12x^3 - 12x^2 - 24x$

$12x\left(x^2 - x - 2\right) = 0$

$x = 0, 2, -1$

Increasing in $(-1, 0) \cup (2, \infty)$

Decreasing in $(-\infty, -1) \cup (0, 2)$

Or

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta; \quad \frac{dy}{d\theta} = -3a\cos^2\theta\sin\theta$$
$$\frac{dy}{dx} = -\cot\theta$$

$$\left.\frac{dy}{dx}\right]_{\theta=\pi/4} = -1$$

$m_T = -1$

$m_N = 1$

$$(x, y) = \left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$$

Equation of tangent $y - \frac{a}{2\sqrt{2}} = -1\left(x - \frac{a}{2\sqrt{2}}\right)$

$$x + y = \frac{a}{\sqrt{2}}$$

Equation of normal

$$y - \frac{a}{2\sqrt{2}} = 1\left(x - \frac{a}{2\sqrt{2}}\right)$$

$x - y = 0$

15. Let $A = \{1, 2, 3, ...9\}$ and R be the relation in A x A defined by (a, b) R (c, d) if a + d = b + c for (a, b), (c, d) in A x A. Prove that R is an equivalence relation. Also obtain the equivalence class [(2, 5)] **[4 Marks]**

Sol. $(a,b)R(c,d)$ on AXA such that

$a + d = b + c$

Reflexivity

$(a,b)R(a,b)$ is true ($\because$ a + b = b + a is always true)

$\therefore$ Q is reflexive

Symmetry

$(a,b)R(c,d)$

$a + d = b + c$

$(c,d)R(a+b)$

$c + b = d + a$

Hence R is symmetric

Transitive

$(a,b)R(c,d) \quad \Rightarrow \quad a + d = b + c$ …………….(1)

$(c,d)R(e,f) \quad \Rightarrow \quad c + f = d + e$ ……………(2)

On adding (1) & (2)

$a + d + c + f = b + c + d + e$

$a + f = b + e$

$\Rightarrow (a,b)R(e,f)$

$\Rightarrow$ R is transitive

$\therefore$ R is equivalence relation.

If $(2,5)R(a,b)$

Then $2 + b = 5 + a$

$\Rightarrow b - a = 3$

$\Rightarrow \{(1, 4),(2,5),(3,6),(4,7),(5,8),(6,9)\}$ are the elements of A x A which will result in relation being equivalent when paired with (2, 5)

16. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2};\ x\in\left(0,\frac{\pi}{4}\right)$ **[4 Marks]**

OR

Prove that $2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$

Sol. $\cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$

$=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}$

OR

To prove that

$2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$

$=2\left[\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right]+\cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)$

$=2\left[\tan^{2}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{40}}\right)\right]+\cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)$

$=2\left[\tan^{-1}\left(\frac{13}{39}\right)\right]+\tan^{-1}\left(\frac{1}{7}\right)$

$=2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$=\tan^{-1}\left(\frac{2(3)}{1-\frac{1}{9}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left[\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right]$

$=\tan^{-1}\left[\frac{21+4}{\frac{28-3}{28}}\right]$

$=\tan^{-1}(1)=\frac{\pi}{4}$

17. If $y=x^{x}$, prove that $\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$ **[4 Marks]**

Sol. $y = x^x$

$\ln(y) = x\ln(x)$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{-1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{1}{x}$

$\Rightarrow \quad y\frac{d^2y}{dx^2} - \frac{dy}{dx} - \frac{y^2}{x} = 0$

or $\quad \frac{d^2y}{dx^2} - \frac{1}{y} - \frac{dy}{dx} - \frac{y}{x} = 0$ Hence proved

18. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls? Given that **[4 Marks]**
(i) the youngest is a girl
(ii) at least one is a girl

Sol. BB BG GG GB
(i) Youngest is a girl
Favourable outcomes are BG, GG
Probability $= \frac{1}{2}$
(ii) At least one is girl
Favourable out comes are BG, GG
Probability = $\frac{1}{3}$

19. Using properties of determinants, prove the following **[4 Marks]**

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1 + x^2 + y^2 + z^2$$

Sol. $$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} x\left(x^2+1\right) & x^2y & x^2z \\ xy^2 & y\left(y^2+1\right) & y^2z \\ xz^2 & yz^2 & z\left(z^2+1\right) \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2+1 & x^2 & x^2 \\ y^2 & y^2+1 & y^2 \\ z^2 & z^2 & z^2+1 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 1+x^2+y^2+z^2 & 1+x^2+y^2+z^2 & 1+x^2+y^2+z^2 \\ y^2 & y^2+1 & y^2 \\ z^2 & z^2 & z^2+1 \end{vmatrix}$$

$$= \left(1 + x^2 + y^2 + z^2\right)\begin{vmatrix} 1 & 1 & 1 \\ y^2 & y^2+1 & y^2 \\ z^2 & z^2 & z^2+1 \end{vmatrix}$$

$C_2 \to C_2 - C_1;\ C_3 \to C_3 - C_1$

$$= \left(1 + x^2 + y^2 + z^2\right)\begin{vmatrix} 1 & 0 & 0 \\ y^2 & 1 & 0 \\ z^2 & 0 & 1 \end{vmatrix}$$

$$= 1 + x^2 + y^2 + z^2$$

20. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$. **[4 Marks]**

Sol. $\mu = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$.

Put $x = \tan\theta$

$\mu = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$= \frac{\theta}{2}$

$v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}\left(\sin 2\theta\right) = 2\theta$

$\frac{dv}{d\theta} = 2$

$\Rightarrow \frac{d\mu}{dv} = \frac{\frac{d\mu}{d\theta}}{\frac{dv}{d\theta}} = \frac{1/2}{2} = \frac{1}{4}$

21. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x\left(2\log x + 1\right)}{\sin y + y\cos y}$, given that $y = \frac{\pi}{2}$ when x = 1. **[4 Marks]**

Sol. $\frac{dy}{dx} = \frac{x\left(2\log x + 1\right)}{\sin y + y\cos y}$

$\left(\sin y + y\cos y\right)dy = \left(2x\log x + x\right)dx$

$\int\left(\sin y + y\cos y\right)dy = \left(2x\log x + x\right)dx$

$\int\left(\sin y + y\cos y\right)dy = 2\int x\log x\,dx + \int x\,dx$

$\int \sin y\,dy + \int y\cos y\,dy = 2\left[\frac{x^2}{2}\log_e x - \int\frac{x^2}{2}.\frac{1}{x}dx\right] + \frac{x^2}{2} + C$

$-\cos y + y\sin y - \int \sin y\,dy = x^2\log_e x - \frac{x^2}{2} + \frac{x^2}{2} + C$

$y\sin y = x^2\log_e x + c$

$x = 1;\ y = \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2}(1) = 1(\log_e 1) + c \Rightarrow c = \frac{\pi}{2}$

$\Rightarrow y \sin y = x^2 \log_e x + \frac{\pi}{2}$

22. Show that lines $\vec{r} = \left(\hat{i} + \hat{j} - \hat{k}\right) + \lambda\left(3\hat{i} - \hat{j}\right)$ and $\vec{r} = \left(4\hat{i} - \hat{k}\right) + \mu\left(2\hat{i} + 3\hat{k}\right)$ intersect. Also find their point of intersection. **[4 Marks]**

Sol. $\vec{r} = \left(\hat{i} + \hat{j} - \hat{k}\right) + \lambda\left(3\hat{i} - \hat{j}\right)$

$\vec{r} = \left(4\hat{i} - \hat{k}\right) + \mu\left(2\hat{i} + 3\hat{k}\right)$

Condition for two lines

$\vec{r} = \vec{a} + \alpha\vec{b}$

and $\vec{r} = \vec{c} + \mu\vec{d}$

to intersect is that $\vec{a} - \vec{c},\ \vec{b}, \vec{d}$ must be coplanar

$\vec{a} - \vec{c} = \hat{i} - \hat{j} - \hat{k} - 4\hat{i} + \hat{k} = -3\hat{i} + \hat{j}$

$\vec{b} = 2\hat{i} + 3\hat{k}$

$\left[\vec{a} - \vec{c}\ \ \vec{b}\ \ \ \vec{d}\right] = 0$

$$\left[\vec{a} - \vec{c}\ \ \vec{b}\ \ \vec{d}\right] = \begin{vmatrix} -3 & 1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 3\left(3 - 3\right) = 0$$

$\Rightarrow$ Lines are intersecting

$\hat{i} + \hat{j} + \hat{k} + \alpha\left(3\hat{i} - \hat{j}\right) = 4\hat{i} - \hat{k} + \mu\left(2\hat{i} + 3\hat{k}\right)$

$\hat{i}\left(1 + 3\alpha\right) + \left(1 - \alpha\right)\hat{j} - \hat{k} = \left(4 + 2\mu\right)\hat{i} + \left(3\mu - 1\right)\hat{k}$

Eqating coefficients of $\hat{i}, \hat{j}, \hat{k}$

$1 + 3\alpha = 4 + 2\mu$

$1 - \alpha = 0 \qquad \Rightarrow \alpha = 1$

$-1 = 3\mu - 1 = \mu = 0$

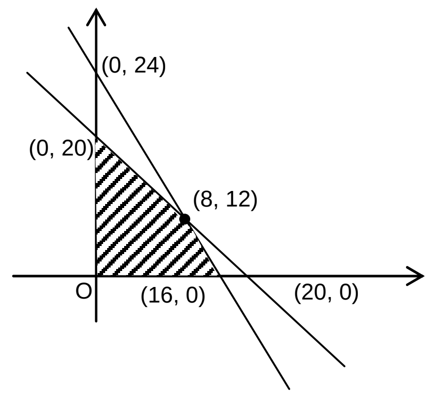
$\alpha = 1$

$\therefore$ Point of intersection is $4\hat{i} - \hat{k}$ or $\left(4, 0, -1\right)$

SECTION C

23. A dealer in rural area wishes to purchase a number of sewing machines. He has only Rs. 5,760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him Rs. 360 and a manually operated sewing machine Rs. 240. He can sell an electronic sewing machine at a profit of Rs. 22 and a manually operated sewing machine at a profit of Rs. 18. Assuming that he can sell at the items that he can buy, how should he invest his money in order to maximize his profit? Make it as a LPP and solve it graphically. **[6 Marks]**

Sol. Let him have x electronic machines and y manual machine



$360x + 240y \le 5760$

$\Rightarrow 36x + 24y \le 576 \Rightarrow 3x + 2y \le 48$

$x + y \le 20$

$\frac{x}{16} + \frac{y}{24} \le 1$

Profit $z = 22x + 18y$

$(0, 20)(16, 0),(8, 12)$

$z = 22 \times 0 + 18 \times 20 = 360$

$z = 22 \times 16 + 18 \times 0 = 352$

$Z = 22(8) + 18(12) = 392$

$\therefore$ Maximum profit happens when x = 8 and y = 12

24. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade. **[6 Marks]**

OR

Form a lot of 15 bulbs which include 5 defective, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of number of defective bulbs. Hence find the mean of the distribution.

Sol. Let E_1 denote the event that lost card is spade.

Let E_2 denote the event that lost card is NOT spade

Let E be the event that 3 cards drawn are spades.

$P(E_1) = \frac{^{13}C_1}{^{52}C_1} = \frac{1}{4} \qquad \Rightarrow P(E_2) = \frac{3}{4}$

$P(E/E_1) = \frac{^{12}C_3}{^{51}C_3}; \;\; P(E/E_2) = \frac{^{13}C_3}{^{51}C_3}$

$P(E_1/E) = \frac{P(E_1 \cap E)}{P(E)} = \frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2)}$

$$= \frac{\frac{1}{4} \times \frac{^{12}C_3}{^{51}C_3}}{\frac{1}{4} \times \frac{^{12}C_3}{^{51}C_3} + \frac{3}{4} \times \frac{^{13}C_3}{^{51}C_3}} = \frac{10}{49}$$

OR

Probability of choosing a defective bulb, $p = \frac{5}{15} = \frac{1}{3}$

$\Rightarrow q = \frac{2}{3}$

n = 4

$P(x = r) = {}^{n}C_r \; p^r \; q^{n-r} = {}^{4}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{4-r} = \frac{^{4}C_r}{3^4}.2^{4-r}$

$P(x = 0) = \frac{16}{81}$

$P(x=1)=\frac{4\times 8}{81}=\frac{32}{81}$

$P(x=2)=\frac{6\times 4}{81}=\frac{24}{81}$

$P(x=3)=\frac{4\times 2}{81}=\frac{8}{81}$

$P(x=4)=\frac{1}{81}$

x = r	r = 0	r = 1	r = 2	r = 3	R = 4
P (X = r)	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

Mean = np = $4\times\frac{1}{3}=\frac{4}{3}$

25. Find the area of the region in the first quadrant enclosed by the x – axis, the line y = x and the circle $x^2+y^2=32$. **[6 Marks]**

Sol. Required area

$=Ar(\Delta OAC)+Area(\Delta CBA)$

$=\frac{1}{2}\times 4\times 4+\int_4^{4\sqrt{2}}\sqrt{32-x^2}\,dx$

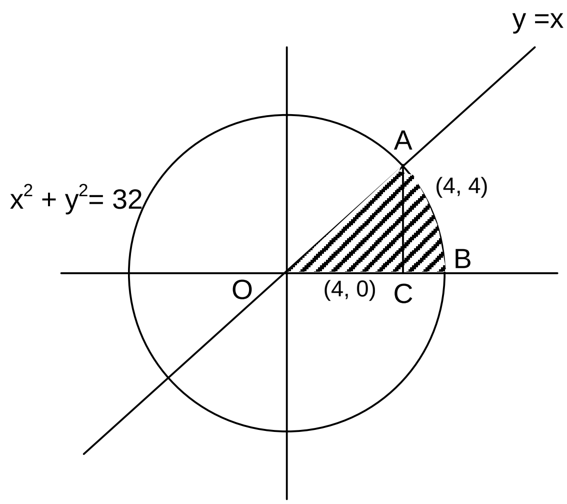
$=8+\left|\frac{x}{2}\sqrt{32-x^2}+16\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right|_4^{4\sqrt{2}}$

$=8+0+16\left(\frac{\pi}{2}\right)-\left(8+16\left(\frac{\pi}{4}\right)\right)$

$=16\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$

$=16\left(\frac{\pi}{4}\right)$

$=4\pi$



26. Find the distance between the point (7, 2, 4) and the plane determined by the points A (2, 5, –3), B (–2, –3, 5) and C (5, 3, –3). **[6 Marks]**

OR

Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ and the plane $\vec{r}.(\hat{i}-\hat{j}+\hat{k})=5$.

Sol. $\begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$

$(x-2)(0+16)-(y-5)(0-24)+(z+3)(8+24)=0$

$16(x-2)+24(y-5)+32(z+3)=0$

$2(x-2)+3(y-5)+4(z+3)=0$

$2x+3y+4z=4+15-12=7$

$2x+3y+4z=7$

$$p=\frac{\left|2(7)+3(2)+4(4)-7\right|}{\sqrt{2^2+3^2+4^2}}=\frac{29}{\sqrt{29}}=\sqrt{29}$$

OR

Solving for line & plane

$$\left\{2\hat{i}-\hat{j}+2\hat{k}+\alpha\left(3\hat{i}+4\hat{j}+2\hat{k}\right)\right\}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$

$$(2+3\lambda)(1)+(-4+4\lambda)(-1)+(2+2\lambda)(1)=5$$

$$2+3\lambda+1-4\lambda+2+2\lambda=5$$

$$\lambda=0$$

$\Rightarrow$ Point of intersection is given by $2\hat{i}-\hat{j}+2\hat{k}\ \ (2,-1,2)$

$\therefore$ Required distance $=\sqrt{(-1-2)^2+(-5+1)^2+(-10-2)^2}$

$= 13$

27. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award Rs. x each, Rs. y each and Rs z each for the three respective values to it 3, 2 and 1 students with a total award money of Rs. 1000. School Q wants to spend Rs. 1500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is Rs. 600, using matrices, find the award money for each value.
Apart from the above three values, suggest one value for awards. **[6 Marks]**

Sol.

$$x+y+z=600$$

$$3x+2y+z=1000$$

$$4x+y+3z=1500$$

Let $A=\begin{bmatrix}1&1&1\\3&2&1\\4&1&3\end{bmatrix};\ x\begin{bmatrix}x\\y\\z\end{bmatrix};\ B=\begin{bmatrix}600\\1000\\1500\end{bmatrix}$

$$AX=B$$

$$|A|=6-1-1(9-4)+1(3-8)=5-5-5=-5$$

$A_{11}=6-1=5$ $\quad A_{12}=(-1)^3(9-4)=-5$ $\quad A_{13}=3-8=-5$

$A_{21}=-2$ $\quad A_{22}=-1$ $\quad A_{23}=(1-4)=3$

$A_{31}=-1$ $\quad A_{32}=-(1-3)=2$ $\quad A_{33}=-1$

adj. $A=\begin{bmatrix}5&-2&-1\\-5&-1&2\\-5&3&-1\end{bmatrix}$ $\qquad A^{-1}=\frac{1}{|A|}\text{adj } A$

$$=\begin{bmatrix}-1&\frac{2}{5}&\frac{1}{5}\\1&\frac{1}{5}&\frac{-2}{5}\\1&\frac{-3}{5}&\frac{1}{5}\end{bmatrix}$$

$$X=A^{-1}B$$

$$=\begin{bmatrix} -1 & \frac{2}{5} & \frac{1}{5} \\ 1 & \frac{1}{5} & \frac{-2}{5} \\ 1 & \frac{-3}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 600 \\ 1000 \\ 1500 \end{bmatrix}$$

$$=\begin{bmatrix} -600+400+300 \\ 600+200-600 \\ 600-600+300 \end{bmatrix}$$

$$=\begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$x=100;\ y=200;\ z=300$

28. Evaluate : $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ **[6 Marks]**

Sol. $I=\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ …………..(i)

Replace x by $\left(\frac{\pi}{2}-x\right)$ Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$I=\int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\cos^4 x+\sin^4 x}dx$ …………………(ii)

Adding (i) & (ii)

$I=\frac{\pi}{4}\int_0^{\pi/2} \frac{\cos x\sin x}{\sin^4 x+\cos^4 x}dx$

$=\frac{\pi}{4}\int_0^{\pi/2} \frac{\tan x\sec^2 x}{\tan^4 x+1}dx=\frac{\pi}{8}\int_0^{\pi/2} \frac{2\tan x\sec^2 x}{\left(\tan^2 x\right)^2+1}dx$

Put $t=\tan^2 x \Rightarrow dt=2\tan x\sec^2 x\,dx$

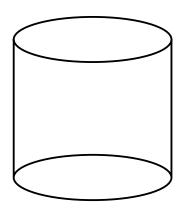
$I=\frac{\pi}{8}\int_0^{\infty} \frac{dt}{t^2+1}=\frac{\pi}{8}\left|\tan^{-1} t\right|_0^{\infty}=\frac{\pi^2}{16}$

29. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area. **[6 Marks]**

Sol. $V=128\pi\ m^3$

$r^2h=128$

$h=\frac{128}{r^2}$



$S=2\pi r^2+2\pi rh$

$=2\pi r^2+2\pi r\left(\frac{128}{r^2}\right)$

$=2\pi r^2+\frac{256\pi}{r^2}$

$$\frac{ds}{dr} = 4\pi r - \frac{256\pi}{r^2}$$

Put $\frac{ds}{dr} = 0 \Rightarrow r^3 = 64 \Rightarrow r = 4$

$$\frac{d^2s}{dr^2} = 4\pi + \frac{512\pi}{r^3} > 0 \text{ for } r = 4$$

Hence r = 4 will give minimum value of S.

$\therefore\ h = \frac{128}{4^2} = 8$

$r = 4;\ h = 8$